2022

MATHEMATICS — HONOURS

Paper : CC-6

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct alternative with proper justification (**1 mark** for correct answer and **1 mark** for justification) : $(1+1)\times 10$

(a) In the ring $(\mathbb{Z}_9, +, \cdot)$, $\overline{6}$ is

(i) a zero divisor
(ii) an invertible element
(iii) not a zero divisor
(iv) an idempotent element.

(b) The ring $(\mathbb{Z}_8, +, \cdot)$ has a subring

(i) $\{\overline{0}, \overline{2}, \overline{4}\}$
(ii) $\{\overline{0}, \overline{2}, \overline{4}, \overline{5}\}$
(iii) $\{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$
(iv) $\{\overline{2}, \overline{4}, \overline{6}\}$

(c) Which one of the following is not a field?

(i) $\mathbb{Z}/2\mathbb{Z}$
(ii) $\mathbb{Z}/3\mathbb{Z}$
(iii) $\mathbb{Z}/4\mathbb{Z}$
(iv) $\mathbb{Z}/5\mathbb{Z}$

(d) The number of solutions of the equation $x^2 - \overline{4}x + \overline{3} = \overline{0}$ in $\mathbb{Z}_{12}$ is

(i) 2
(ii) 4
(iii) 6
(iv) 12

(e) For any two coprime numbers m, n, the kernel of the ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$ defined by $f(x) = (\overline{x}, \overline{x})$ is

(i) $mn\mathbb{Z}$
(ii) $m\mathbb{Z}$
(iii) $n\mathbb{Z}$
(iv) $\mathbb{Z}$

(f) Let $\mathbb{Q}$ be the ring of rational numbers and $\mathbb{R}$ be the ring of real numbers. Let $\mathbb{Z}$ be the set of all integers, then

(i) $\mathbb{Z}$ is an ideal of $\mathbb{Q}$ but $\mathbb{Q}$ is not an ideal of $\mathbb{R}$

(ii) $\mathbb{Z}$ is not an ideal of $\mathbb{Q}$ and $\mathbb{Q}$ is not an ideal of $\mathbb{R}$

(iii) $\mathbb{Z}$ is not an ideal of $\mathbb{R}$ but $\mathbb{Q}$ is an ideal of $\mathbb{R}$

(iv) $\mathbb{Z}$ is an ideal of $\mathbb{Q}$ and $\mathbb{Q}$ is an ideal of $\mathbb{R}$

(g) Let V be the real vector space of all 3×3 real matrices and W be the sub-space of V consisting of all symmetric matrices. Then the dimension of W is

(i) 9 (ii) 6

(iii) 3 (iv) 8

(h) Let V be the three-dimensional vector space over the field $\mathbb{Z}_3$. The number of elements of V is

(i) 3 (ii) 9

(iii) 27 (iv) 81

(i) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_2, 0)$. Then

(i) $\dim \ker T = 2$ (ii) $\dim \operatorname{Im} T = 2$

(iii) $\ker T = \operatorname{Im} T$ (iv) $\ker T \subsetneq \operatorname{Im} T$

(j) The eigenvalues of $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$ are

(i) 1, 2, 2 (ii) 1, −1, 2

(iii) 3, 3, 8 (iv) 3, 6, 8.

Unit – I

Answer *any five* questions.

2. (a) (i) Prove that the characteristic of an integral domain is either zero or a prime number.

(ii) Prove that in a finite ring R with unity 1_R, $a.b = 1_R$ for some $a, b \in R$ implies $b.a = 1_R$.

2+3

(b) (i) Let R be a ring and $Z(R) = \{x \in R : xr = rx \text{ for all } r \in R\}$. Prove that $Z(R)$ is a subring of R.

(ii) Let $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Prove that the subring $\mathbb{Q}[\sqrt{2}]$ of $\mathbb{R}$ is a subfield of $\mathbb{R}$.

2+3

(c) (i) Give an example of a ring R and a proper ideal M of R which is a maximal ideal but not a prime ideal.

(ii) Let R be a ring in which every element is idempotent element. Prove that R is a commutative ring. 2+3

(d) If R is a commutative ring with identity and M an ideal of R, show that R/M is a field if and only if M is a maximal ideal of R. 5

(e) Let $C[0, 1]$ be the ring of all real valued continuous functions on the closed interval $[0, 1]$. Show that the set $S = \left\{ f \in C[0, 1] : f\left(\frac{1}{2}\right) = 0 \right\}$ is a maximal ideal of $C[0, 1]$. Also prove that $C[0, 1]/S \simeq \mathbb{R}$ where $\mathbb{R}$ is the field of real numbers. 3+2

(f) Let R be a ring and ρ be a ring congruence on R. Then prove that the ρ-equivalence class containing 0 is a subring of R. Is this an ideal of R? Justify your answer. 3+2

(g) Let R be the ring $\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ and $\varphi : R \to \mathbb{Z}$ is defined by $\varphi \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a - b$. Show that φ is a ring homomorphism. Determine ker φ. Show that the ring $R/\ker \varphi$ is isomorphic to $\mathbb{Z}$. 2+1+2

(h) Let $I = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid 5 \text{ devides } n\}$. Show that I is a prime ideal of $\mathbb{Z} \times \mathbb{Z}$. Is it a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$? Justify your answer. 2+3

Unit – II

Answer *any four* questions.

3. (a) Prove that there exists a basis for each finite dimensional vector space. 5

(b) (i) Find the dimension of the subspace S of $\mathbb{R}^3$ where $S = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0\}$.

(ii) Find the co-ordinate vector of $\alpha = (1, 3, 1)$ relative to the ordered basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $\mathbb{R}^3$. 3+2

(c) Let V_1 and V_2 be two vector spaces over a field F and let V_1 be finite dimensional. If $f : V_1 \to V_2$ be a linear mapping, then prove that nullity of f + rank of f = dim V_1. 5

(d) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $T(x, y) = (x + y, y, y)$. Find the matrix representation of T with respect to the basis $B_1 = \{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$ and $B_2 = \{(1, 1, 1), (0, 1, 0), (0, 0, 1)\}$ of $\mathbb{R}^3$. 5

(e) Prove that two finite dimensional vector spaces V and W over a field F are isomorphic if and only if dim V = dim W. 5

(f) (i) Use Cayley-Hamilton theorem to compute A^{-1} where $A=\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}$.

(ii) Show that the eigenvalues of a Hermitian matrix are all real. 3+2

(g) (i) Let P be a real orthogonal matrix with det $P = -1$. Prove that -1 is an eigenvalue of P.

(ii) If λ be an eigenvalue of an $n \times n$ matrix A, then prove that λ^2 is an eigenvalue of A^2. 3+2